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## DOMAIN DECOMPOSITION ALGORITHMS FOR FIRST-ORDER SYSTEM LEAST SQUARES METHODS

**Luca F. Pavarino**

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*Institute for Computer Applications in Science and Engineering  
NASA Langley Research Center  
Hampton, VA 23681-0001*

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*National Aeronautics and  
Space Administration*

*Langley Research Center  
Hampton, Virginia 23681-0001*



# Domain Decomposition Algorithms for First-Order System Least Squares Methods

*Luca F. Pavarino\**

Dipartimento di Matematica  
Università di Pavia  
Via Abbiategrasso 209  
27100 Pavia, Italy  
pavarino@dragon.ian.pv.cnr.it.

## Abstract

Least squares methods based on first-order systems have been recently proposed and analyzed for second-order elliptic equations and systems. They produce symmetric and positive definite discrete systems by using standard finite element spaces, which are not required to satisfy the inf-sup condition. In this paper, several domain decomposition algorithms for these first-order least squares methods are studied. Some representative overlapping and substructuring algorithms are considered in their additive and multiplicative variants. The theoretical and numerical results obtained show that the classical convergence bounds (on the iteration operator) for standard Galerkin discretizations are also valid for least squares methods.

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# 1 Introduction

Least squares methods have been proposed in recent years for second-order elliptic problems, Stokes and Navier-Stokes equations; see Chang [10], Bochev and Gunzburger [2], Pehlivanov, Carey, and Lazarov [15], Cai, Lazarov, Manteuffel, and McCormick [5], Cai, Manteuffel, and McCormick [7], Bramble, Lazarov, and Pasciak [3], Bramble and Pasciak [4], Carey, Pehlivanov, and Vassilevski [8], Cai, Manteuffel, and McCormick [6], Bochev, Cai, Manteuffel, and McCormick [1], and the references therein.

Among the possible approaches, we follow here the one introduced in the very recent works of Pehlivanov, Carey, and Lazarov [15] and Cai, Manteuffel, and McCormick [7]. The second-order elliptic problem is rewritten as a first-order system and a least squares functional is introduced. The resulting discrete minimization problem is associated with a bilinear form which is continuous and elliptic on an appropriate space. Therefore, the inf-sup condition is avoided and standard finite element spaces can be used. The resulting linear system is symmetric, positive definite and has condition number of the same order as standard Galerkin finite element stiffness matrices,  $O(1/h^2)$ . An interesting alternative approach by Bramble, Lazarov, and Pasciak [3] is based on replacing one of the  $L^2$ -terms in the least squares functional by a discrete  $H^{-1}$ -norm. We will not consider here such an alternative.

The goal of this paper is to extend to these least squares methods some of the classical domain decomposition algorithms which have been successfully employed for standard Galerkin finite elements. We show that optimal and quasi-optimal convergence bounds follow easily from the standard Galerkin case. Therefore, domain decomposition provides highly parallel and scalable solvers also for first-order system least square discretizations. An overview of domain decomposition methods can be found in the review papers by Chan and Mathew [9], Dryja, Smith, and Widlund [11], Dryja and Widlund [13], and Le Tallec [14].

This paper is organized as follows. In the next section, we briefly review the first-order system least squares methodology and the main results from [7]. In Section 3, we introduce and analyze our domain decomposition algorithms: overlapping additive Schwarz methods (with coupled and uncoupled subspaces; see 3.1), overlapping multiplicative Schwarz methods (3.2), and an iterative substructuring method (3.3). In Section 4, we present numerical results in the plane that confirm the theoretical bounds obtained.

## 2 First-Order System Least Squares

We consider the following second-order elliptic problem on a bounded domain  $\Omega \subset R^2$  or  $R^3$

$$\begin{cases} -\nabla \cdot (A\nabla p) + Xp = f & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ \mathbf{n} \cdot A\nabla p = 0 & \text{on } \Gamma_N. \end{cases} \quad (1)$$

Here  $A$  is a symmetric and uniformly positive definite matrix with entries in  $L^\infty(\Omega)$ ,  $X$  is a first-order linear differential operator,  $\Gamma_D \cup \Gamma_N = \partial\Omega$ , and  $\mathbf{n}$  is the outward normal unit vector to  $\Gamma_N$ .

Defining the new flux variable  $\mathbf{u} = -A\nabla p$ , the system (1) can be rewritten as a first-order system:

$$\begin{cases} \mathbf{u} + A\nabla p = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + Xp = f & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} = 0 & \text{on } \Gamma_N, \end{cases} \quad (2)$$

This system can be extended to the equivalent system

$$\begin{cases} \mathbf{u} + A\nabla p = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} + Xp = f & \text{in } \Omega, \\ \nabla \times A^{-1}\mathbf{u} = \mathbf{0} & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} = 0 & \text{on } \Gamma_N, \\ \gamma_\tau(A^{-1}\mathbf{u}) = 0 & \text{on } \Gamma_D, \end{cases} \quad (3)$$

where  $\nabla \times$  is the curl operator (in two dimensions  $\nabla \times \mathbf{u} = 0$  means  $\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = 0$ ) and  $\gamma_\tau \mathbf{u} = \mathbf{u} \times \mathbf{n}$  (in two dimensions  $\gamma_\tau \mathbf{u} = \mathbf{u} \cdot \tau$ ).

The following least squares functionals,  $G_0$  for the system (2) and  $G$  for the augmented system (3), were studied in [5] ([15] for the case  $X = 0$ ) and [7], respectively:

$$G_0(\mathbf{v}, q; f) = \|\mathbf{v} + A\nabla q\|_{L^2(\Omega)}^2 + \|\nabla \cdot \mathbf{v} + Xq - f\|_{L^2(\Omega)}^2 \quad (4)$$

$\forall (\mathbf{v}, q) \in \mathbf{W}_0(\text{div}; \Omega) \times V$ , and

$$G(\mathbf{v}, q; f) = \|\mathbf{v} + A\nabla q\|_{L^2(\Omega)}^2 + \|\nabla \cdot \mathbf{v} + Xq - f\|_{L^2(\Omega)}^2 + \|\nabla \times (A^{-1}\mathbf{v})\|_{L^2(\Omega)}^2 \quad (5)$$

$\forall(\mathbf{v}, q) \in \mathbf{W} \times V$ .

Here the functional spaces are defined as

$$\begin{aligned}\mathbf{W}_0(\operatorname{div}; \Omega) &= \{\mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega) : \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \Gamma_N\}, \\ \mathbf{W}_0(\operatorname{curl} A; \Omega) &= \{\mathbf{v} \in \mathbf{H}(\operatorname{curl} A; \Omega) : \gamma_\tau(A^{-1}\mathbf{v}) = 0 \text{ on } \Gamma_D\}, \\ \mathbf{W} &= \mathbf{W}_0(\operatorname{div}; \Omega) \cap \mathbf{W}_0(\operatorname{curl} A; \Omega), \\ V &= \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_D\}.\end{aligned}$$

The least squares minimization problems for (2) and (3) are, respectively:  
Find  $(\mathbf{u}, p) \in \mathbf{W}_0(\operatorname{div}; \Omega) \times V$  such that

$$G_0(\mathbf{u}, p; f) = \inf_{(\mathbf{v}, q) \in \mathbf{W}_0(\operatorname{div}; \Omega) \times V} G_0(\mathbf{v}, q; f); \quad (6)$$

Find  $(\mathbf{u}, p) \in \mathbf{W} \times V$  such that

$$G(\mathbf{u}, p; f) = \inf_{(\mathbf{v}, q) \in \mathbf{W} \times V} G(\mathbf{v}, q; f). \quad (7)$$

Simple calculations show that the associated variational problems are, respectively:

Find  $(\mathbf{u}, p) \in \mathbf{W}_0(\operatorname{div}; \Omega) \times V$  such that

$$a_0(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q) \quad \forall(\mathbf{v}, q) \in \mathbf{W}_0(\operatorname{div}; \Omega) \times V; \quad (8)$$

Find  $(\mathbf{u}, p) \in \mathbf{W} \times V$  such that

$$a(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q) \quad \forall(\mathbf{v}, q) \in \mathbf{W} \times V. \quad (9)$$

Here the bilinear forms are

$$\begin{aligned}a_0(\mathbf{u}, p; \mathbf{v}, q) &= (\mathbf{u} + A\nabla p, \mathbf{v} + A\nabla q)_{L^2} + (\nabla \cdot \mathbf{u} + Xp, \nabla \cdot \mathbf{v} + Xq)_{L^2}, \\ a(\mathbf{u}, p; \mathbf{v}, q) &= a_0(\mathbf{u}, p; \mathbf{v}, q) + (\nabla \times (A^{-1}\mathbf{u}), \nabla \times (A^{-1}\mathbf{v}))_{L^2}\end{aligned}$$

and the right-hand side is

$$F(\mathbf{v}, q) = (f, \nabla \cdot \mathbf{v} + Xq)_{L^2}.$$

In [5], it was proved that  $a_0(\mathbf{v}, q; \mathbf{v}, q)$  is equivalent to (continuous and elliptic with respect to) the  $\mathbf{H}(\operatorname{div}; \Omega) \times H^1(\Omega)$  norm on  $\mathbf{W}_0(\operatorname{div}; \Omega) \times V$ , under

the assumption that a Poincaré-Friedrichs inequality holds for  $p$  (denoted by assumption A0). For the case  $X = 0$ , this was already proved in [15].

In [7], it was proved that  $a(\mathbf{v}, q; \mathbf{v}, q)$  is equivalent to the  $[\mathbf{H}(\text{div}; \Omega) \cap \mathbf{H}(\text{curl}; \Omega)] \times H^1(\Omega)$  norm on  $\mathbf{W} \times V$ , under the same assumption A0. Moreover, under three additional technical assumptions denoted by A1, A2, A3, it is proven in [7] that  $a(\mathbf{v}, q; \mathbf{v}, q)$  is equivalent to the  $H^1(\Omega)^{d+1}$  norm on  $\mathbf{W} \times V$  ( $d = 2$  or  $3$ ):

**Theorem 1** *Let  $b(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{u}, \mathbf{v})_{H^1(\Omega)^d} + (p, q)_{H^1(\Omega)}$  be the bilinear form associated with the  $H^1(\Omega)^{d+1}$  norm.*

*If the assumptions A0-A3 of [7] are verified, then there exist positive constants  $\alpha$  and  $\beta$  such that*

$$\alpha b(\mathbf{v}, q; \mathbf{v}, q) \leq a(\mathbf{v}, q; \mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in \mathbf{W} \times V,$$

and

$$a(\mathbf{u}, p; \mathbf{v}, q) \leq \beta b(\mathbf{u}, p; \mathbf{u}, p)^{1/2} b(\mathbf{v}, q; \mathbf{v}, q)^{1/2}$$

$$\forall (\mathbf{u}, p), (\mathbf{v}, q) \in \mathbf{W} \times V.$$

Because of the equivalence of  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$ , from now on we will concentrate on the variational problem (9) associated with the augmented system (3).

We introduce a triangulation  $\tau_h$  of  $\Omega$  and associated finite element subspaces  $\mathbf{W}^h \times V^h \subset \mathbf{W} \times V$ . We then obtain a finite element discretization of (9):

Find  $(\mathbf{u}_h, p_h) \in \mathbf{W}^h \times V^h$  such that

$$a(\mathbf{u}_h, p_h; \mathbf{v}_h, q_h) = F(\mathbf{v}_h, q_h) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{W}^h \times V^h. \quad (10)$$

For simplicity, we consider continuous piecewise linear finite elements:

$$\mathbf{W}^h = \{\mathbf{v} \in C^0(\Omega)^d : v_k|_T \in P_1(T), \forall T \in \tau_h, \mathbf{v} \in \mathbf{W}\} = W_1^h \times W_2^h \times W_3^h,$$

$$V^h = \{q \in C^0(\Omega) : q|_K \in P_1(K), \forall K \in \tau_h, q \in V\}.$$

and the subscript  $h$  for discrete functions will be dropped in the rest of the paper.

Error estimates and results on the conditioning of the resulting stiffness matrix can be found in [5] (in [15] for the case  $X = 0$ ).

Upon choosing a basis in  $\mathbf{W}^h$  and  $V_h$ , the discrete problem (10) is turned into a linear system of equations  $Ax = b$ . We are going to solve such system iteratively using domain decomposition techniques.



### 3 Domain Decomposition Algorithms

We will introduce and analyze our domain decomposition algorithms in the Schwarz framework, which has been very successful for standard Galerkin finite elements, see [9], [11], [12], [13]. We illustrate the main ideas on algorithms that are representative of the main classes of domain decomposition (additive, multiplicative, overlapping, iterative substructuring). The same analysis can be applied to the many other algorithms which have been proposed and analyzed for the standard scalar case.

We suppose that the domain  $\Omega$  is first triangulated by a coarse finite element triangulation  $\tau_H$  consisting of  $N$  subdomains  $\Omega_i$  of diameter  $H$ . The fine triangulation  $\tau_h$  is a refinement of  $\tau_H$ . For simplicity, we suppose that each subdomain is the image under a smooth map of a reference cube.

#### 3.1 Overlapping Additive Schwarz Methods

Each subdomain  $\Omega_i$  is extended to a larger subdomain  $\Omega'_i$ , consisting of all elements of  $\tau_h$  within a distance  $\delta$  from  $\Omega_i$  ( $0 < \delta < H$ ).

Each scalar component of our finite element space  $\mathbf{W}^h \times V^h$  is decomposed as in the standard scalar case:

$$W_1^h = \sum_{i=1}^N W_{1,i}^h, \quad W_2^h = \sum_{i=1}^N W_{2,i}^h, \quad W_3^h = \sum_{i=1}^N W_{3,i}^h, \quad V^h = \sum_{i=1}^N V_i^h,$$

where

$$W_{k,i}^h = \{u \in W_k^h : \text{support}(u) \subset \Omega'_i\}, \quad k = 1, 2, 3, \\ V_i^h = \{u \in V^h : \text{support}(u) \subset \Omega'_i\}.$$

For each scalar component, a global coarse finite element space is associated with the coarse triangulation  $\tau_H$ :

$$W_{k,0}^h = W_k^H = \{u \in W_k^h : u \text{ is trilinear on each subdomain } \Omega_i\}, \quad k = 1, 2, 3,$$

$$V_0^h = V^H = \{p \in V^h : p \text{ is trilinear on each subdomain } \Omega_i\}.$$

A first additive method is defined by the following decomposition of the discrete space, which maintains the local and coarse coupling between the

different scalar components:

$$\mathbf{W}^h \times V^h = \sum_{i=0}^N \mathbf{W}_i^h \times V_i^h.$$

The local spaces are

$$\mathbf{W}_i^h \times V_i^h = W_{1,i}^h \times W_{2,i}^h \times W_{3,i}^h \times V_i^h \quad i = 1, 2, \dots, N$$

and the coarse space is

$$\mathbf{W}_0^h \times V_0^h = \mathbf{W}^H \times V^H = W_1^H \times W_2^H \times W_3^H \times V^H.$$

We define the local projection operators  $P_i : \mathbf{W}^h \times V^h \rightarrow \mathbf{W}_i^h \times V_i^h$  by

$$a(P_i(\mathbf{u}, p); \mathbf{v}, q) = a(\mathbf{u}, p; \mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in \mathbf{W}_i^h \times V_i^h,$$

and the coarse projection operator  $P_0 : \mathbf{W}^h \times V^h \rightarrow \mathbf{W}_0^h \times V_0^h$  by

$$a(P_0(\mathbf{u}, p); \mathbf{v}, q) = a(\mathbf{u}, p; \mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in \mathbf{W}_0^h \times V_0^h.$$

It is easy to see that the matrix form of the local projections is  $P_i = R_i^T A_i^{-1} R_i A$ , where the  $R_i(\mathbf{e}_k) = \begin{cases} 1 & \text{if } \mathbf{e}_k \in \Omega'_i \\ 0 & \text{otherwise} \end{cases}$  are the restriction matrices selecting only the unknowns in  $\Omega'_i$  for each component and the  $A_i = R_i A R_i^T$  are the stiffness matrices of local Dirichlet problems. Analogously,  $P_0 = R_H^T A_H^{-1} R_H A$ , where  $R_H^T$  is the standard piecewise linear interpolation matrix from the coarse grid  $\tau_H$  to the fine grid  $\tau_h$ , for each component, and  $A_H = R_H A R_H^T$  is the coarse grid discretization of our problem (9). Let

$$P_{add1} = \sum_{i=0}^N P_i.$$

The original discrete problem is then equivalent to the preconditioned problem

$$P_{add1}(\mathbf{u}, p) = \mathbf{g}_{add1}$$

where  $\mathbf{g} = \sum_{i=0}^N P_i(\mathbf{u}, p)$ ; see Chan and Mathew [9]. In matrix form, this problem can be written as  $M^{-1}Ax = M^{-1}b$ , where the preconditioner is

$M^{-1} = \sum_{i=1}^N R_i^T A_i^{-1} R_i + R_H^T A_H^{-1} R_H$ . An optimal convergence bound for this algorithm is given in Theorem 2.

A second additive method is obtained by dropping the coupling between the different scalar components of  $\mathbf{u}$  and  $p$ . Uncoupled local spaces are now defined by

$$\begin{aligned}\mathbf{W}_{1,i}^h &= W_{1,i}^h \times \{0\} \times \{0\} \times \{0\}, \\ \mathbf{W}_{2,i}^h &= \{0\} \times W_{2,i}^h \times \{0\} \times \{0\}, \\ \mathbf{W}_{3,i}^h &= \{0\} \times \{0\} \times W_{3,i}^h \times \{0\}, \\ \mathbf{V}_i^h &= \{0\} \times \{0\} \times \{0\} \times V_i^h,\end{aligned}$$

and the coarse spaces by

$$\begin{aligned}\mathbf{W}_1^H &= \mathbf{W}_{1,0}^h = W_{1,0}^h \times \{0\} \times \{0\} \times \{0\}, \\ \mathbf{W}_2^H &= \mathbf{W}_{2,0}^h = \{0\} \times W_{2,0}^h \times \{0\} \times \{0\}, \\ \mathbf{W}_3^H &= \mathbf{W}_{3,0}^h = \{0\} \times \{0\} \times W_{3,0}^h \times \{0\}, \\ \mathbf{V}^H &= \mathbf{V}_0^h = \{0\} \times \{0\} \times \{0\} \times V_0^h.\end{aligned}$$

We then have the following decomposition

$$\begin{aligned}\mathbf{W}^h \times V^h &= \sum_{i=1}^N \mathbf{W}_{1,i}^h + \sum_{i=1}^N \mathbf{W}_{2,i}^h + \sum_{i=1}^N \mathbf{W}_{3,i}^h + \sum_{i=1}^N \mathbf{V}_i^h + \mathbf{W}_1^H + \mathbf{W}_2^H + \mathbf{W}_3^H + \mathbf{V}^H \\ &= \sum_{k=1}^3 \sum_{i=0}^N \mathbf{W}_{k,i}^h + \sum_{i=0}^N \mathbf{V}_i^h.\end{aligned}$$

As before, we define projections  $P_{k,i} : \mathbf{W}^h \times V^h \rightarrow \mathbf{W}_{k,i}^h$ ,  $k = 1, 2, 3$ ,  $i = 0, 1, \dots, N$  and  $P_{4,i} : \mathbf{W}^h \times V^h \rightarrow \mathbf{V}_i^h$ ,  $i = 0, 1, \dots, N$ , and the additive operator

$$P_{add2} = \sum_{k=1}^3 \sum_{i=0}^N P_{k,i} + \sum_{i=0}^N P_{4,i}.$$

We note that this algorithm can equivalently be defined by the same choice of subspaces as for  $P_{add1}$  but using the bilinear form  $b(\cdot, \cdot)$  (introduced in Theorem 1) instead of  $a(\cdot, \cdot)$  in the definition of the projections. In fact this uncoupled preconditioner corresponds to applying four identical copies of a scalar preconditioner to each scalar component. An optimal bound also holds for this algorithm.

**Theorem 2** *There exists a positive constant  $C$  independent of  $h, H$  and  $\delta$  such that*

$$\text{cond}(P) \leq C(1 + \frac{H}{\delta}),$$

where  $P = P_{add1}$  or  $P = P_{add2}$ .

*Proof.* An upper bound on the spectrum of  $P$  is standard, since each point of  $\Omega$  belongs to a fixed number of extended subdomains independent of  $N$  (for example, for  $\delta < H/2$  each point belongs to at most four (in 2D) or eight (in 3D) extended subdomains). A lower bound is obtained by classical Schwarz analysis.

For  $P = P_{add1}$ , since we use exact projections, the theorem is equivalent to the following partition property (see Dryja and Widlund [13] or Chan and Mathew [9]):

There exists a constant  $C_0$  such that  $\forall (\mathbf{u}, p) \in \mathbf{W}^h \times V^h$ , there exists a decomposition  $(\mathbf{u}, p) = \sum_{i=0}^N (\mathbf{u}_i, p_i)$ , with  $(\mathbf{u}_i, p_i) \in \mathbf{W}_i^h \times V_i^h$  such that

$$\sum_{i=0}^N a(\mathbf{u}_i, p_i; \mathbf{u}_i, p_i) \leq C_0^2 a(\mathbf{u}, p; \mathbf{u}, p).$$

By the equivalence of Theorem 1, this inequality is equivalent to

$$\sum_{i=0}^N |(\mathbf{u}_i, p_i)|_{(H^1)^{d+1}}^2 \leq C_0^2 |(\mathbf{u}, p)|_{(H^1)^{d+1}}^2,$$

which is a direct consequence of the scalar result proven by Dryja and Widlund [13]:

$$\sum_{i=0}^N |u_i^k|_{H^1}^2 \leq C_0^2 |u^k|_{H^1}^2, \quad \sum_{i=0}^N |p_i|_{H^1}^2 \leq C_0^2 |p|_{H^1}^2,$$

with  $C_0^2 = C(1 + \frac{H}{\delta})$ .

For  $P = P_{add2}$ , since the subspaces are the same but we use inexact projections defined by  $b(\cdot, \cdot)$  instead of  $a(\cdot, \cdot)$ , we need only to show that there exists a constant  $\omega$  such that  $a(\mathbf{u}, p; \mathbf{u}, p) \leq \omega b(\mathbf{u}, p; \mathbf{u}, p) \quad \forall (\mathbf{u}, p) \in \mathbf{W}_i^h \times V_i^h, \quad i = 0, 1, \dots, N$  (see Dryja and Widlund [12]). This follows immediately from the equivalence of  $a$  and  $b$ .

### 3.2 Overlapping Multiplicative Schwarz Methods

By using the same coupled local and coarse spaces as in the additive algorithm  $P_{add1}$ , we can define a multiplicative operator:

$$P_{mult} = I - (I - P_N) \cdots (I - P_1)(I - P_0).$$

The multiplicative algorithm consists in solving the nonsymmetric system

$$P_{mult}(\mathbf{u}, p) = \mathbf{g}_{mult}$$

by an iterative method such as GMRES.

We can also define a symmetrized multiplicative operator

$$P_{mults} = I - (I - P_0) \cdots (I - P_{N-1})(I - P_N)(I - P_{N-1}) \cdots (I - P_0)$$

and a symmetrized algorithm, consisting in solving the symmetric system

$$P_{mults}(\mathbf{u}, p) = \mathbf{g}_{mults}$$

by an iterative method like CG.

**Theorem 3** *There exists a positive constant  $C$  independent of  $h, H$  and  $\delta$  such that*

$$\text{cond}(P_{mults}) \leq C(1 + \frac{H}{\delta}).$$

The proof is again based on the extension of the scalar result (see, for example, Chan and Mathew [9]) by using the equivalence of Theorem 1. Analogously, multiplicative versions of  $P_{add2}$  could be built using uncoupled local and coarse spaces.

### 3.3 An Iterative Substructuring Method

For a complete and detailed analysis of this class of methods, we refer to Dryja, Smith and Widlund [11]. Here we consider only a simple representative of this class, namely the analog of Algorithm 6.2 in [11], which is vertex-based and has a standard coarse space. For simplicity, we only consider the uncoupled additive version.

The standard first step of nonoverlapping methods is the elimination of the variables interior to each subdomain (at least implicitly). We then work with the Schur complement  $S = K_{BB} - K_{IB}^T K_{II}^{-1} K_{IB}$  of the stiffness matrix

$$K = \begin{pmatrix} K_{II} & K_{IB} \\ K_{IB}^T & K_{BB} \end{pmatrix}.$$

The reduced linear system with  $S$  involves only variables on the interface  $\Gamma = \cup \partial\Omega_i \setminus \Gamma_D$ . When solving with a preconditioned iterative method, we need only the action of  $S$  on a given vector and there is no need to assemble  $S$  explicitly.

In the Schwarz framework, working with  $S$  corresponds to working with the discrete harmonic subspace  $\tilde{\mathbf{W}}^h \times \tilde{V}^h$  of the original space  $\mathbf{W}^h \times V^h$ . Local spaces are associated with the geometric objects (faces  $F_i$ , edges  $E_i$  and vertices  $v_i$ ) forming the interface  $\Gamma$ . Each scalar space is decomposed as

$$\tilde{W}_k^h = \sum_{F_i} \tilde{W}_{k,F_i}^h + \sum_{E_i} \tilde{W}_{k,E_i}^h + \sum_{v_i} \tilde{W}_{k,v_i}^h, \quad k = 1, 2, 3,$$

and

$$\tilde{V}^h = \sum_{F_i} \tilde{V}_{F_i}^h + \sum_{E_i} \tilde{V}_{E_i}^h + \sum_{v_i} \tilde{V}_{v_i}^h.$$

Here, for example,  $\tilde{W}_{k,F_i}^h = \{u \in \tilde{W}^h : u = 0 \text{ on } \Gamma_h - F_{i,h}\}$ , where  $\Gamma_h$  and  $F_{i,h}$  are the set of nodes on  $\Gamma$  and  $F_i$  respectively. The other spaces are defined analogously. As for the overlapping case, we then embed these scalar spaces in our product space  $\tilde{\mathbf{W}}^h \times \tilde{V}^h$ : for example,  $\tilde{\mathbf{W}}_{1,F_i}^h = \tilde{W}_{1,F_i}^h \times \{0\} \times \{0\} \times \{0\}$ . As a coarse space, we consider the discrete harmonic subspace of the same coarse space used for  $P_{add2}$ , i.e.,  $\tilde{\mathbf{W}}_1^H + \tilde{\mathbf{W}}_2^H + \tilde{\mathbf{W}}_3^H + \tilde{\mathbf{V}}^H$ . We obtain the following decomposition

$$\begin{aligned} \tilde{\mathbf{W}}^h \times \tilde{V}^h &= \sum_{k=1}^3 \left( \sum_{F_i} \tilde{\mathbf{W}}_{k,F_i}^h + \sum_{E_i} \tilde{\mathbf{W}}_{k,E_i}^h + \sum_{v_i} \tilde{\mathbf{W}}_{k,v_i}^h + \tilde{\mathbf{W}}_k^H \right) \\ &\quad + \sum_{F_i} \tilde{\mathbf{V}}_{F_i}^h + \sum_{E_i} \tilde{\mathbf{V}}_{E_i}^h + \sum_{v_i} \tilde{\mathbf{V}}_{v_i}^h + \tilde{\mathbf{V}}^H. \end{aligned}$$

By defining as before projection operators into the subspaces, we form the additive operator

$$P_{is} = \sum_{k=1}^4 \left( \sum_{F_i} P_{k,F_i} + \sum_{E_i} P_{k,E_i} + \sum_{v_i} P_{k,v_i} + P_{k,0} \right),$$

where again for  $k = 4$  the projections are into the  $\tilde{\mathbf{V}}_i^h$  spaces.

**Theorem 4** *There exists a positive constant  $C$  independent of  $h$  and  $H$  such that*

$$\text{cond}(P_{is}) \leq C(1 + \log(H/h))^2.$$

As before, the proof is based on the extension of the scalar result (see Dryja, Smith and Widlund [11], Theorem 6.2) by using the equivalence of Theorem 1.

## 4 Numerical Results

In this section, we report the results of numerical experiments which confirm the optimal convergence bounds obtained in the previous sections. All the results have been obtained with Matlab 4.2 running on Sun Sparcstations. The model problem considered is the standard Poisson equation ( $A = I, X = 0$ ) on the unit square, with  $p = 0$  on  $\Gamma_D = \partial\Omega$  and  $\gamma_\tau \mathbf{u} = 0$  on  $\partial\Omega$  (i.e.  $u_1 = 0$  on  $\{y = 0\}$  and  $\{y = 1\}$ ;  $u_2 = 0$  on  $\{x = 0\}$  and  $\{x = 1\}$ ). The right-hand side  $f$  is chosen such that we have  $p(x, y) = \sin(\pi x)\sin(\pi y)$  as exact solution.  $\Omega$  is decomposed into a regular grid of  $N$  square subdomains, with  $N$  varying from  $2 \times 2$  to  $8 \times 8$ . The fine grid mesh size  $h$  varies from  $1/32$  to  $1/128$ .

The Krylov method used for all the symmetric problems is PCG, while we use GMRES for the nonsymmetric problem with  $P_{mult}$ . The initial guess is always zero and the stopping criterion is  $\|r_k\|_2/\|r_0\|_2 < 10^{-6}$ , where  $r_k$  is the residual at step  $k$ .

The local and coarse problems involved in the application of the preconditioners are always solved directly. For each method, we report the number of iterations and Lanczos-based estimates of the condition number and the extreme eigenvalues (except for the multiplicative algorithm, where we report the average convergence factor instead).

*Overlapping additive methods.* We have first studied the coupled method  $P_{add1}$  with fixed minimal overlap size  $\delta = h$ . The mesh size  $h$  is decreased while the number of subdomains  $N$  is increased proportionally, so that the subdomain size  $H/h = 16$  is kept constant ( $H = 1/\sqrt{N}$ ). The results are reported in Table 1 and clearly show a constant condition number  $\text{cond}(P_{add1}) = \lambda_{max}/\lambda_{min}$ , for problem sizes from 3007 ( $N = 4$ ) to 48895 ( $N = 64$ ).

Table 1:  $P_{add1}$ : Overlapping Additive Schwarz with fixed overlap size  $\delta = h$ .

$N$	$h^{-1}$	$iter.$	$cond(P_{add1})$	$\lambda_{max}$	$\lambda_{min}$
4	32	16	11.2172	4.0048	0.3570
9	48	19	12.1787	4.0068	0.3290
16	64	20	11.9775	4.0050	0.3343
25	80	20	11.1689	4.0052	0.3586
36	96	21	12.5450	4.0044	0.3192
49	112	20	11.9944	4.0050	0.3339
64	128	21	12.5500	4.0047	0.3191

Table 2:  $P_{add1}$ : Overlapping Additive Schwarz with fixed number of subdomains  $N = 64$ .

$\delta$	$h^{-1}$	$iter.$	$cond(P_{add1})$	$\lambda_{max}$	$\lambda_{min}$
$h$	128	21	12.5500	4.0047	0.3191
$2h$	128	17	7.1316	4.0307	0.5651
$3h$	128	16	5.5769	4.0765	0.7309
$4h$	128	15	4.9540	4.1396	0.8356
$5h$	128	15	4.6460	4.2170	0.9076
$6h$	128	15	4.5125	4.3054	0.9541
$7h$	128	16	4.5859	4.4018	0.9598



Table 3:  $P_{add2}$  : Overlapping Additive Schwarz with fixed overlap size  $\delta = h$ .

$N$	$h^{-1}$	$iter.$	$cond(P_{add2})$	$\lambda_{max}$	$\lambda_{min}$
4	32	17	10.3521	4.0050	0.3868
9	48	20	12.6290	4.0051	0.3171
16	64	20	11.9811	4.0051	0.3342
25	80	21	11.3821	4.0052	0.3518
36	96	21	12.5458	4.0043	0.3191
49	112	20	11.9997	4.0052	0.3337
64	128	21	12.5261	4.0047	0.3197

Table 4:  $P_{add2}$  : Overlapping Additive Schwarz with fixed number of subdomains  $N = 64$ .

$\delta$	$h^{-1}$	$iter.$	$cond(P_{add2})$	$\lambda_{max}$	$\lambda_{min}$
$h$	128	21	12.5261	4.0047	0.3197
$2h$	128	17	7.1206	4.0315	0.5661
$3h$	128	16	5.5513	4.0777	0.7345
$4h$	128	16	5.3850	4.1442	0.7695
$5h$	128	16	5.4545	4.2233	0.7742
$6h$	128	16	5.5306	4.3158	0.7803
$7h$	128	16	5.6176	4.4297	0.7885

Table 5:  $P_{mult}$  and  $P_{mults}$ : Overlapping Multiplicative Schwarz with fixed overlap size  $\delta = h$ .

$N$	$h^{-1}$	multiplicative (GMRES)		symmetrized multiplicative (CG)			
		$iter.$	$\rho = (r_i/r_0)^{1/i}$	$iter.$	$cond(P_{mults})$	$\lambda_{max}$	$\lambda_{min}$
4	32	8	0.1847	7	1.8576	0.9994	0.5379
9	48	7	0.1433	6	1.7398	0.9999	0.5749
16	64	6	0.1233	6	1.7600	0.9999	0.5681
25	80	6	0.1102	6	1.6810	0.9999	0.5948
36	96	6	0.1021	6	1.6940	0.9999	0.5902
49	112	6	0.0952	6	1.6661	0.9999	0.6001
64	128	5	0.0849	6	1.7308	0.9999	0.6079

In Table 2, we fix the mesh size ( $h = 1/128$ ) and the decomposition ( $N = 64$ ) and we vary the overlap size  $\delta$  from  $h$  to  $7h$ . As in the scalar case, the condition number  $cond(P_{add1})$  improves as  $\delta$  increases, because of  $\lambda_{min}$  being closer to unity. For large overlap, the improvement becomes negligible or negative, because of the growth of  $\lambda_{max}$ .

The same sets of results for the uncoupled method  $P_{add2}$  are reported in Table 3 and Table 4, respectively. For this simple model problem, the uncoupled method is only slightly worse than the coupled one, in terms of iteration count (some condition numbers are almost the same or even better for  $P_{add2}$ ). We point out that although  $A = I$ , eliminating diffusive coupling between the flux components, there is still coupling between the flux variables and  $p$ , so the strong performance of  $P_{add2}$  is encouraging.

*Overlapping multiplicative methods.* In Table 5, we compare the multiplicative method  $P_{mult}$  accelerated with GMRES and the symmetrized multiplicative method  $P_{mults}$  accelerated with CG. We consider the two methods with minimal overlap and constant subdomain size. Since  $P_{mult}$  is nonsymmetric, we report the average convergence factor  $\rho = (r_i/r_0)^{1/i}$  instead of the condition number. Even if the symmetrized version is approximately twice as expensive as the standard one, the number of iterations is almost the same for the two methods. Therefore, the symmetrized version is less efficient on this simple problem.

Table 6:  $P_{is}$ : Iterative Substructuring .

$N$	$h^{-1}$	$iter.$	$cond(P_{is})$	$\lambda_{max}$	$\lambda_{min}$
4	32	9	3.4035	1.5691	0.4610
9	48	17	7.8812	1.8497	0.2347
16	64	18	7.8543	1.7962	0.2287
25	80	18	8.5822	1.8864	0.2198
36	96	19	9.4115	1.8511	0.1966
49	112	18	8.6646	1.8939	0.2185
64	128	19	9.6532	1.8617	0.1928

*Iterative substructuring.* Table 6 shows the results for the iterative substructuring methods  $P_{is}$  with fixed subdomain size. They clearly show a constant bound for the condition number and the number of iterations.

## 5 Conclusions

In this paper, some domain decomposition algorithms have been introduced for the discrete systems arising from first-order system least squares methods applied to second-order elliptic problems. These recently proposed methods allow the use of standard finite element spaces, which are not required to satisfy the inf-sup condition.

The analysis of the domain decomposition algorithms follows from analogous results for the standard Galerkin case and the equivalence between the bilinear form associated with the least squares functional and the  $H^1(\Omega)^{d+1}$  norm.

Optimal convergence bounds have been proven for overlapping algorithms (additive, multiplicative, coupled, uncoupled versions), while quasi-optimal bounds have been proven for iterative substructuring algorithms. Numerical experiments on a simple model problem confirm these bounds.

Future work will investigate the performance of these algorithms for problems with convection and for elliptic systems.

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